

Classification of wormhole singularities

Birational Geometry Seminar, UCLA

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Motivation

- Let $0 < q < m$ be coprime integers and let $(Q \in Y)$ be a (c.q.s) cyclic quotient singularity $\frac{1}{m}(1, q)$. Kollár and Shepherd-Barron introduced the notion of P-resolutions to study the deformation space of cyclic quotient singularities.

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- Wormhole singularities were introduced in 2021 by Urzúa and Vilches to study this particular wormhole phenomenon in the Kollar–Shepherd-Barron–Alexeev (KSBA) compactification of the moduli space of surfaces of general type. A wormhole singularity is a cyclic quotient singularity which admits at least two different extremal P-resolutions.

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- The classification of these singularities has been open.

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My main contribution is a definition! This definition is the notion of **coherent graph of a framed triangulated polygon**. It provides us with a framework where the problem of classification can be solved, and an alternative proof of the HTU-theorem on the maximum number of extremal P-resolutions that is much more straightforward.

Cyclic quotient singularities

Let $m > 1$ be an integer, and ξ a m -th primitive root of 1. Let $a, b > 0$ be integers coprime to m , and consider the action of $\mathbb{Z}/m\mathbb{Z}$ on \mathbb{C}^2 given by

$$T: \mathbb{C}^2 \rightarrow \mathbb{C}^2, (x, y) \mapsto (\xi^a x, \xi^b y)$$

Definition 1.

A cyclic quotient singularity (c.q.s) is a singularity isomorphic to the germ at 0 of the quotient $\mathbb{C}^2/\langle T \rangle$ for some integers m, a and b . We will denote the singularity as $\frac{1}{m}(a, b)$.

It is enough to restrict our attention to the case $\frac{1}{m}(1, q)$. Its resolution is constructed with toric methods and it is summarized as follows:

Theorem 2.

Let $p \in X$ be a c.q.s $\frac{1}{m}(1, q)$, where X is a neighborhood of p and $\phi: Y \rightarrow X$ its minimal resolution. Then $\text{Exc}(\phi)$ is a collection of rational curves E_1, \dots, E_r such that $E_i \cdot E_{i+1} = 1$ for $i \in \{1, \dots, r-1\}$, $E_i^2 = -e_i$ and $E_i \cdot E_j = 0$ for any other case, where

$$\frac{m}{q} = [e_1, \dots, e_r] = e_1 - \frac{1}{e_2 - \frac{1}{\dots \frac{1}{e_r}}}$$

is its Hirzebruch-Jung continued fraction (each e_i is an integer greater than 1).

T-singularities

Definition 3.

A T-singularity is a quotient singularity which admits a \mathbb{Q} -Gorenstein one parameter smoothing.

Proposition 1.

T-singularities are either ADE or c.q.s $\frac{1}{dn^2}(1, dna - 1)$ with $0 < a < n$, $d \geq 1$ and $\gcd(n, a) = 1$. When $d = 1$, we call them Wahl singularities.

Definition 4.

A T-chain is the exceptional divisor of the minimal resolution of a T-singularity (non-ADE). For a Wahl singularity, we call it a W-chain.

A result of Wahl gives us an algorithm to obtain all (non-ADE) T-singularities. Furthermore, it tells us how a T-singularity can be recognized from its minimal resolution.

T-singularities

Proposition 2 (Algorithm for T-chains).

For any (non-ADE) T-singularity $\frac{1}{dn^2}(1, dna - 1)$, we have:

- (i) If $n = 2$, then $[4]$ and $[3, 2, \dots, 2, 3]$ where the number of 2's in this expression is $d - 2$, are T-chains.
- (ii) If $[b_1, \dots, b_r]$ is a T-chain, then so are $[2, b_1, \dots, b_{r-1}, b_r + 1]$ and $[b_1 + 1, b_2, \dots, b_r, 2]$.
- (iii) Any T-chain can be obtained by starting with one of the T-chains in (i) and iterating the steps described in (ii).

The T-chain associated with $\frac{1}{dn^2}(1, dna - 1)$ will be denoted as $[d \binom{n}{a}]$. The operations in (ii) will be called \mathcal{L} -operations and \mathcal{R} -operations, respectively. The proof of this result also shows that

$$\left[d \binom{2n - a}{n} \right] \xleftarrow{\mathcal{L}} \left[d \binom{n}{a} \right] \quad y \quad \left[d \binom{n}{a} \right] \xrightarrow{\mathcal{R}} \left[d \binom{n + a}{a} \right]$$

T-singularities

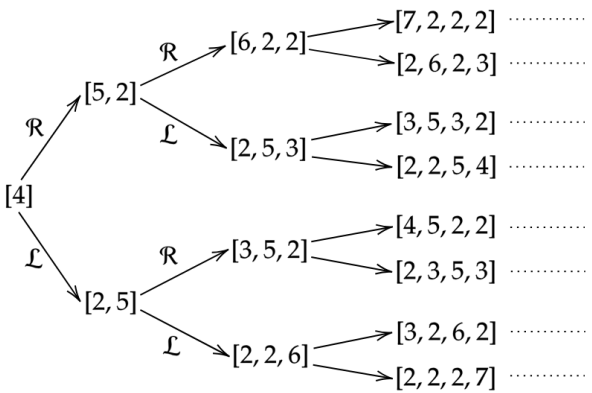


Figure: W-chains.

P-resolutions

Definition 5.

Let $P \in Y$ be a c.q.s. A P-resolution of $P \in Y$ is a **partial resolution** $f: X \rightarrow Y$ over P such that X only have T-singularities and K_X is ample relative to f .

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Definition 6.

Let $\frac{m}{q} = [e_1, \dots, e_r]$ be a H-J continued fraction. Its dual fraction is

$$\frac{m}{m-q} := [b_1, \dots, b_s].$$

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From the Riemenschneider's dot diagram, if we write

$$\frac{m}{q} = [\underbrace{2, \dots, 2}_{a_1}, b_1, \underbrace{2, \dots, 2}_{a_2}, b_2, \dots, \underbrace{2, \dots, 2}_{a_{c-1}}, b_{c-1}, \underbrace{2, \dots, 2}_{a_c}]$$

where $a_i \geq 0$ and $b_i \geq 3$ for all i , then

$$\frac{m}{m-q} = [a_1 + 2, \underbrace{2, \dots, 2}_{b_1-3}, a_2 + 3, \underbrace{2, \dots, 2}_{b_2-3}, a_3 + 3, \dots, a_{c-1} + 3, \underbrace{2, \dots, 2}_{b_{c-1}-3}, a_c + 2]$$

P-resolutions

Each fraction together with its dual induce a zero continued fraction. Namely, if

$\frac{m}{q} = [e_1, \dots, e_r]$ and $\frac{m}{m-q} = [b_1, \dots, b_s]$ then:

$$[b_s, \dots, b_1, 1, e_1, \dots, e_r] = 0$$

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Key relation 1: There is a well-known one-to-one correspondence between zero continued fractions and triangulated polygons with a hidden index.

Theorem 7.

Let (b_1, \dots, b_s) be a sequence of positive integers. Then $[b_1, \dots, b_s] = 0$ if and only if there exists b_0 such that (b_0, b_1, \dots, b_s) is the vector of indices of some triangulated $(s + 1)$ -gon.

The number b_0 is called the hidden index of the triangulation associated with the zero continued fraction $[b_1, \dots, b_s] = 0$.

Given a triangulated $(s + 1)$ -gon with vertices P_0, \dots, P_s , one defines the index of a vertex P_i as

$$v_i := (\text{number of diagonals from the vertex } P_i) + 1.$$

Definition 8.

We say that a triangulated polygon \mathcal{P} is a **framed triangulated polygon** if we specify a hidden index.

P-resolutions

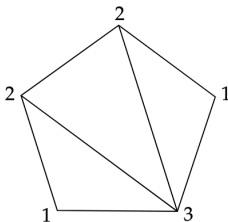


Figure: Triangulated pentagon.

We have the 0-fractions: $[3, 1, 2, 2]$, $[1, 2, 2, 1]$, $[2, 2, 1, 3]$, $[2, 1, 3, 1]$, and $[1, 3, 1, 2]$.

P-resolutions

Lemma 9.

Let \mathcal{P} be a convex polygon with $s + 1$ sides. The indices from the vertices of \mathcal{P} will be taken mod $s + 1$. Consider $[b_1, \dots, b_s] = 0$ for a triangulation of \mathcal{P} , then

1. $b_0 + b_1 \cdots + b_s = 3(s - 1)$.
2. At least two b_i must be equal to 1. Furthermore, for $s \geq 3$, the entries equal to 1 cannot be in consecutive positions.

P-resolutions

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2. At least two b_i must be equal to 1. Furthermore, for $s \geq 3$, the entries equal to 1 cannot be in consecutive positions.

By the relation (1), the number b_0 is completely determined by the linear relation

$$b_0 := 3(s - 1) - \sum_{i=1}^s b_i.$$

Therefore, we can write the zero continued fraction $[b_1, \dots, b_s]$ as $[b_1, \dots, b_s \mid b_0]$. The latter notation is called extended zero chain.

Extremal P-resolutions

Definition 10.

Let $0 < q < m$ be coprime integers, and let $(Q \in Y)$ be a cyclic quotient singularity $\frac{1}{m}(1, q)$. An extremal P-resolution of $(Q \in Y)$ is a partial resolution $f: (C \subset X) \rightarrow (Q \in Y)$, such that X has only Wahl singularities, there is one exceptional curve C and isomorphic to \mathbb{P}^1 , and K_X is ample relative to f .

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Theorem 11 (Hacking-Tevelev-Urzúa, 2013).

Let $0 < q < m$ be coprime integers. If $\frac{m}{m-q} = [k_1, \dots, k_s]$ where $k_i \geq 2$, then there is a bijection between extremal P-resolutions associated with the cyclic quotient singularity $\frac{1}{m}(1, q)$ and pairs $1 \leq \alpha < \beta \leq s$ such that

$$[k_1, \dots, k_{\alpha-1}, k_{\alpha} - 1, k_{\alpha+1}, \dots, k_{\beta-1}, k_{\beta} - 1, k_{\beta+1}, \dots, k_s] = 0.$$

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Definition 12.

A sequence $\{b_1, \dots, b_s\}$, $b_i > 1$ is a WW-sequence if there exists $1 \leq \alpha < \beta \leq s$ such that

$$[b_1, \dots, b_{\alpha} - 1, \dots, b_{\beta} - 1, \dots, b_s] = 0.$$

The numbers α and β are called the indices of the WW-sequence, and we say that the zero continued fraction $0 = [b_1, \dots, b_{\alpha} - 1, \dots, b_{\beta} - 1, \dots, b_s]$ is a WW-decomposition of the Hirzebruch-Jung continued fraction $[b_1, \dots, b_s]$.

Wormholes

Definition 13.

A wormhole singularity is a cyclic quotient singularity $\frac{1}{m}(1, q)$ with $0 < q < m$ coprime integers, which admits at least two distinct extremal P -resolutions. Equivalently, the Hirzebruch-Jung continued fraction of $\frac{m}{m-q}$ has at least two different WW-decompositions.

Examples of wormhole singularities

Example 14.

$$\frac{m}{q} := \frac{900}{241} = [4, 4, 5, 2, 2, 3, 2, 2].$$

Then

$$\frac{m}{m-q} = [2, 2, 3, 2, 3, 2, 2, 5, 4].$$

We have the extremal P-resolutions

$$\left[\binom{11}{3} \right] - (1) - \left[\binom{19}{5} \right] \quad \text{associated with} \quad [2, 2, 3, 2, 2, 2, 1, 5, 4] = 0.$$

$$\left[\binom{26}{7} \right] - (1) - \left[\binom{4}{1} \right] \quad \text{associated with} \quad [2, 2, 3, 2, 3, 1, 2, 4, 4] = 0.$$

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Lemma 15.

Let $\{k_1, \dots, k_s\}$ be a WW-sequence. Assume that $0 = [b_1, \dots, b_s]$ and $0 = [b'_1, \dots, b'_s]$ are WW-decompositions of $[k_1, \dots, k_s]$. If $[b_1, \dots, b_s \mid v_0]$ and $[b'_1, \dots, b'_s \mid v'_0]$ are the corresponding extended zero chains, then $v_0 = v'_0$. This number v_0 is called the WW-index of the WW-sequence $\{k_1, \dots, k_s\}$.

Basic wormhole singularities

Definition 16.

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Example 17.

$$\frac{m}{q} := \frac{800}{289} = [3, 5, 2, 2, 7, 2, 2, 2].$$

Then

$$\frac{m}{m-q} := [2, 3, 2, 2, 5, 2, 2, 2, 2, 5].$$

We have the extremal P-resolutions:

- $-(3) - \left[\binom{17}{4} \right]$ associated with $[1, 3, 2, 2, 5, 1, 2, 2, 2, 5] = 0$.
- $\left[\binom{5}{2} \right] - (2) - \left[\binom{5}{1} \right]$ associated with $[2, 3, 1, 2, 5, 2, 2, 2, 1, 5] = 0$.

Computing the hidden index b_0 , we get that $b_0 = 2$ for this pair.

$$0 = [1, 3, 2, 2, 5, 1, 2, 2, 2, 5 \mid 2]$$

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1. Reduce the classification of wormhole singularities to the classification of basic wormhole singularities (HTU-algorithm).
2. Define basic wormhole triangulations, and notice that we can classify basic wormhole singularities by classifying basic wormhole triangulations.
3. Candidates to basic wormhole triangulations: framed accordion triangulations.
4. Introduce the coherent graph of a framed triangulated polygon and its properties.
5. Define a standard frame of an accordion triangulation. Then define standard family of accordion triangulations and coherent rotation of diagonals.
6. Classify framed accordion triangulations with a standard frame that are basic wormhole triangulations.
7. Justify how we can change the frame from the classification in the step above to classify all basic wormhole triangulations.

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If \mathcal{P} is a basic wormhole triangulation associated with a basic wormhole singularity, then any other triangulation associated with other WW-decomposition of the same basic wormhole singularity is called a companion of \mathcal{P} .

Main results

The main result is that the wormhole condition is equivalent to the consistency of an explicit system of linear relations naturally associated with accordion triangulations.

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Theorem 19 (N., 2025).

Let $n \geq 2$ be a integer, and let \mathcal{P} be a framed accordion triangulation with a standard frame and weights x_1, \dots, x_n . Then \mathcal{P} is a basic wormhole triangulation with a standard frame if and only if there exists an integer $1 \leq m \leq n - 1$ such that the system $S_0 \cup S_m$ of $2n$ linear equations

$$\begin{aligned} S_0 : y_i &= x_{n-i \pmod n} - k_i^{(0)} && \text{for } 1 \leq i \leq n, \\ S_m : y_i &= x_{(n-i)+m \pmod n} - k_i^{(m)} && \text{for } 1 \leq i \leq n, \end{aligned}$$

is consistent; where the numbers $k_i^{(0)}$ and $k_i^{(m)}$ are equal to 3 for all i , except at four specific positions determined by n and m , where they are equal to 1.

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is consistent; where the numbers $k_i^{(0)}$ and $k_i^{(m)}$ are equal to 3 for all i , except at four specific positions determined by n and m , where they are equal to 1.

Explicitly, the numbers $k_i^{(0)}$ and $k_i^{(m)}$ are given as follows:

- If n is odd:** $k_{\frac{n-1}{2}}^{(0)} = 1$, $k_n^{(0)} = 1$, $k_{\frac{n-1}{2} + \lceil \frac{m}{2} \rceil \pmod n}^{(m)} = 1$, $k_{n + \lfloor \frac{m}{2} \rfloor \pmod n}^{(m)} = 1$,
 otherwise $k_i^{(0)} = 3$ and $k_i^{(m)} = 3$.
- If n is even:** $k_{\frac{n}{2}}^{(0)} = 1$, $k_n^{(0)} = 1$, $k_{\frac{n}{2} + \lfloor \frac{m}{2} \rfloor \pmod n}^{(m)} = 1$, $k_{n + \lfloor \frac{m}{2} \rfloor \pmod n}^{(m)} = 1$, otherwise
 $k_i^{(0)} = 3$ and $k_i^{(m)} = 3$.

Main results

The consistency of this system of linear relations depends on explicit numerical conditions involving n and m . If $S_0 \cup S_m$ is consistent, we give a parametric solution depending on $\gcd(n, m)$ -parameters and explain how to recover a parametric family of basic wormhole triangulations with a standard frame from this parametric solution.

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Corollary 20 (N. ,2025).

We give a 3-step constructive algorithm to obtain all basic wormhole triangulations with a fixed number of weights.

Input: An integer $n \geq 2$.

Output: All parametric families of basic wormhole triangulations with n weights.

Accordion triangulation

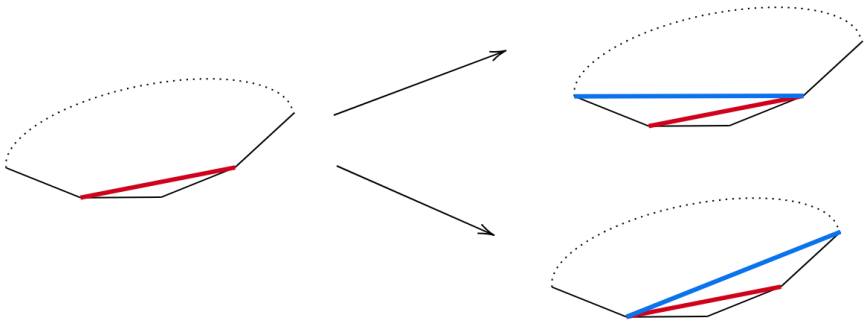


Figure: Construction of an accordion triangulation.

Accordion triangulation

Lemma 22.

Let \mathcal{P} be a triangulated polygon. Then \mathcal{P} has exactly two index 1 vertices if and only if \mathcal{P} is an accordion triangulation.

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Proposition 3.

Let \mathcal{P} be a basic wormhole triangulation. Then \mathcal{P} is a framed accordion triangulation.

Coherent graph

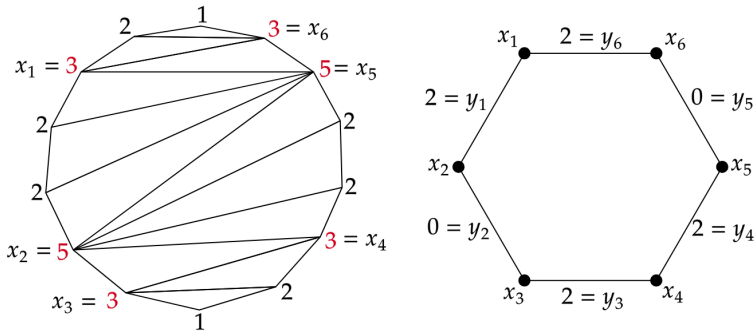


Figure: An example of a framed accordion triangulation \mathcal{P} and its coherent graph $G_{\mathcal{P}}$.

Note that $\mathcal{P} = [1, 2, 3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5 \mid 3]$,
 $\mathcal{P}_1 = [2, 3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5, 3 \mid 1]$, and $\mathcal{P}_2 = [3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5, 3, 1 \mid 2]$
 have the same coherent graph.

Coherent graph of framed accordion triangulations

The coherent graph is defined for any framed triangulated $(s + 1)$ -gon, however, the most powerful applications of these graphs are for framed accordion triangulations. In this case, the coherent graph is naturally endowed with a very explicit set of linear relations.

Coherent graph of framed accordion triangulations

The coherent graph is defined for any framed triangulated $(s + 1)$ -gon, however, the most powerful applications of these graphs are for framed accordion triangulations. In this case, the coherent graph is naturally endowed with a very explicit set of linear relations.

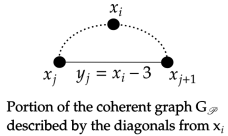
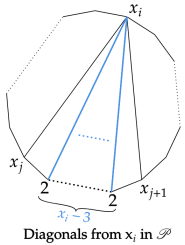
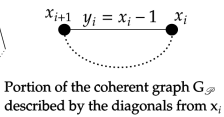
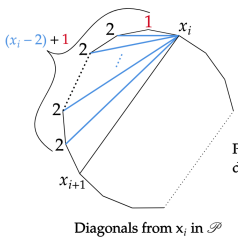


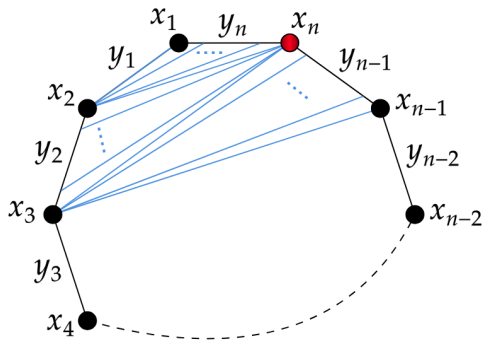
Figure: Geometry of accordion triangulation.

Coherent graph of framed accordion triangulations

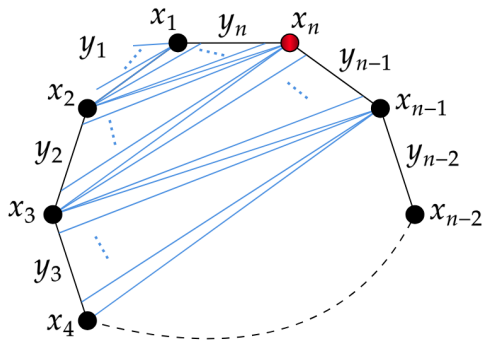
Definition 24.

We say that diagonals from the weight x_i go to the pair $(x_{j \pmod n}, x_{j+1 \pmod n})$ if all the diagonals in \mathcal{P} from x_i fit between x_j and x_{j+1} .

Coherent graph of framed accordion triangulations, $x_n \mapsto (x_2, x_3)$



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Coherent graph of framed accordion triangulations

Lemma 26.

Let \mathcal{P} be a framed accordion triangulation and let $G_{\mathcal{P}} = ([x_1, \dots, x_{n-1} \mid x_n], (y_1, \dots, y_n))$ be its coherent graph. If the diagonals from the weight x_i go to the pair $(x_{j \pmod{n}}, x_{j+1 \pmod{n}})$ for some pair of indices $1 \leq i, j \leq n$, then there is a linear system of n relations associated with the graph $G_{\mathcal{P}}$. Specifically,

$$y_{\ell} = x_{\ell_{i,j}} - k_{\ell_{i,j}} \quad \text{for } 1 \leq \ell \leq n, \quad (1)$$

where each weight in \mathcal{P} is exactly one of the numbers $x_{\ell_{i,j}}$ for some appropriate ℓ , and $k_{\ell_{i,j}}$ is equal to 3 for all ℓ , except for two special values of ℓ for which $k_{\ell_{i,j}}$ is equal to 1.

Coherent graph of framed accordion triangulations

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By the alternative description of accordion triangulations the system of linear relations above does not depend on the choice of the weight x_i , it depends only on \mathcal{P} .

Coherent graph of framed accordion triangulations

Definition 27.

Let \mathcal{P} and $\tilde{\mathcal{P}}$ be framed triangulated polygons. We say that the coherent graphs

$$G_{\mathcal{P}} = ([x_1, \dots, x_{n-1} \mid x_n], (y_1, \dots, y_n)), \quad G_{\tilde{\mathcal{P}}} = ([\tilde{x}_1, \dots, \tilde{x}_{m-1} \mid \tilde{x}_m], (\tilde{y}_1, \dots, \tilde{y}_m))$$

are **equal** if and only if $n = m$, $x_i = \tilde{x}_i$ and $y_i = \tilde{y}_i$ for $1 \leq i \leq n$.

Framed triangulation and a companion

Example 29.

Consider the framed triangulated polygons $\mathcal{P} = [1, 2, 3, 2, 2, 5, 3, 1, 2, 3, 2, 2, 5 \mid 3]$ and $\tilde{\mathcal{P}} = [2, 2, 3, 1, 2, 5, 3, 2, 2, 3, 1, 2, 5 \mid 3]$, see Figure. Note that \mathcal{P} and $\tilde{\mathcal{P}}$ correspond to WW-decompositions of the WW-sequence $\{2, 2, 3, 2, 2, 5, 3, 2, 2, 3, 2, 2\}$ with WW-index $x_6 = 3$.

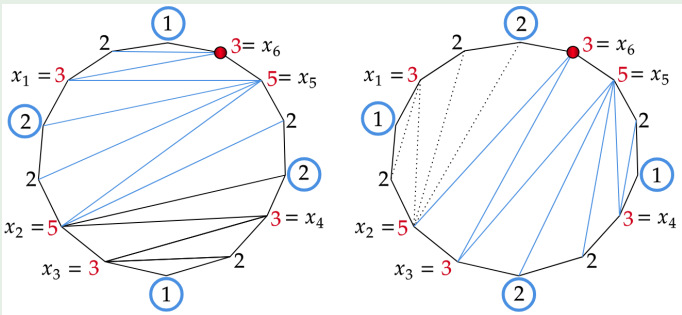


Figure: Framed triangulated polygon and a companion.

The triangulations $\tilde{\mathcal{P}}$ and \mathcal{P} have the same indices in the same positions except at 4 positions where we are swapping a pair of index 1 vertices with a pair of index 2 vertices.

Coherent graph of framed accordion triangulations

Recall that accordion triangulations do not come with a frame. However, we can take a frame that is almost canonical.

Definition 30.

Let \mathcal{P} be an accordion triangulation. A standard frame for \mathcal{P} is a frame of \mathcal{P} where the hidden vertex is a weight of type (I) and the first entry of the extended zero chain is 1.

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Definition 31.

Let $n \geq 2$ be an integer, let x_1, \dots, x_n be n integer variables with the restriction that $x_i \geq 3$. Let $\mathcal{P}_{x_1, \dots, x_n}^m$ be the unique accordion triangulation with weights x_1, \dots, x_n (in that order) such that the diagonals from x_n go to the pair $(x_{m \pmod n}, x_{m+1 \pmod n})$ for $0 \leq m \leq n - 1$. The frame of $\mathcal{P}_{x_1, \dots, x_n}^m$ is the one given by the ordering x_1, \dots, x_n with hidden index x_n . We denote by \mathcal{P}_m the family of all triangulations $\mathcal{P}_{x_1, \dots, x_n}^m$. The family \mathcal{P}_0 is called the standard family of accordion triangulations, whereas the family \mathcal{P}_m for $1 \leq m \leq n - 1$ is called the coherent family obtained after m -coherent rotations of diagonals from the standard family \mathcal{P}_0 . We say that $\mathcal{P}_{x_1, \dots, x_n}^m$ in the family \mathcal{P}_m is obtained from $\mathcal{P}_{x_1, \dots, x_n}^0$ via m -coherent rotation of diagonals. For any element $\mathcal{P}_{x_1, \dots, x_n}^m$ in the family \mathcal{P}_m , we denote by S_m the system of linear relations associated with the coherent graph $G_{\mathcal{P}_{x_1, \dots, x_n}^m}$ where $0 \leq m \leq n - 1$.

Coherent graph of framed accordion triangulations

Given $n \geq 2$ an integer, and an integer $0 \leq m \leq n - 1$ it is straightforward to explicitly compute the linear system of n relations associated with $G_{\mathcal{P}_{x_1, \dots, x_n}^m}$. Specifically,

$$S_m: y_i = x_{(n-i)+m(\bmod n)} - k_i^{(m)} \quad \text{for } 1 \leq i \leq n,$$

where the numbers $k_i^{(m)}$ are given as follows:

- **If n is odd:** $k_{\frac{n-1}{2} + \lceil \frac{m}{2} \rceil}^{(m)} = 1$, $k_{n + \lfloor \frac{m}{2} \rfloor}^{(m)} = 1$, otherwise $k_i^{(m)} = 3$,
- **If n is even:** $k_{\frac{n}{2} + \lceil \frac{m}{2} \rceil}^{(m)} = 1$, $k_{n + \lfloor \frac{m}{2} \rfloor}^{(m)} = 1$, otherwise $k_i^{(m)} = 3$,

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ represent the ceiling and floor functions respectively.

Example: Classification of basic wormhole triangulations with 2 weights.

There is only one coherent rotation of diagonals of $G_{\mathcal{P}^0_{x_1, x_2}}$ given by the coherent graph $G_{\mathcal{P}^1_{x_1, x_2}}$ in Figure 9.

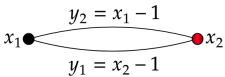


Figure: Coherent graph $G_{\mathcal{P}^0_{x_1, x_2}}$.

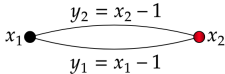
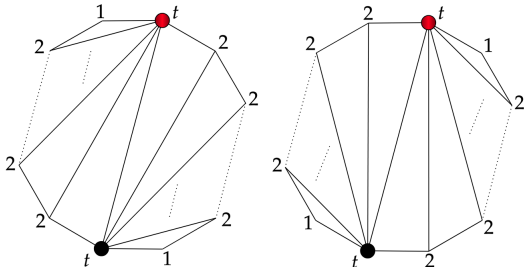


Figure: Coherent graph $G_{\mathcal{P}^1_{x_1, x_2}}$.

By Main theorem, we have that $\mathcal{P}^1_{x_1, x_2} = (\mathcal{P}^0_{x_1, x_2})^\vee$ if and only if $x_1 - 1 = x_2 - 1$. Thus $(x_1, x_2) = (t, t)$ for $t \geq 3$.



Example: Classification of basic wormhole triangulations with 3 weights.

There are 2 coherent rotations of diagonals of $G_{\mathcal{P}^0_{x_1, x_2, x_3}}$.

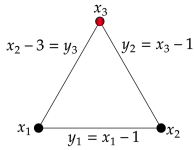
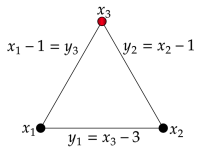
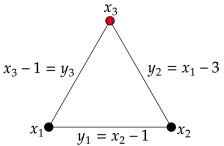


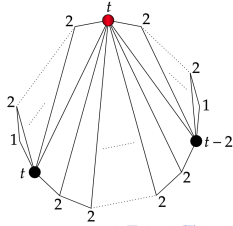
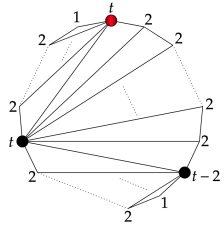
Figure: Coherent graph $G_{\mathcal{P}^0_{x_1, x_2}}$.

Figure: Coherent rotations of diagonals.

- $\mathcal{P}^1_{x_1, x_2, x_3} = (\mathcal{P}^0_{x_1, x_2, x_3})^\vee$ if and only if we have the relations:

$$x_3 - 1 = x_1 - 1, \quad x_2 - 1 = x_3 - 3, \quad x_1 - 3 = x_2 - 1.$$

Thus $(x_1, x_2, x_3) = (t, t - 2, t)$ for $t \geq 5$.



Example: Classification of basic wormhole triangulations with 3 weights.

2. $\mathcal{P}_{x_1, x_2, x_3}^2 = (\mathcal{P}_{x_1, x_2, x_3}^0)^\vee$ if and only if we have the relations:

$$x_3 - 1 = x_2 - 3, \quad x_2 - 1 = x_1 - 1, \quad x_1 - 3 = x_3 - 1.$$

Thus $(x_1, x_2, x_3) = (t, t, t - 2)$ for $t \geq 5$.

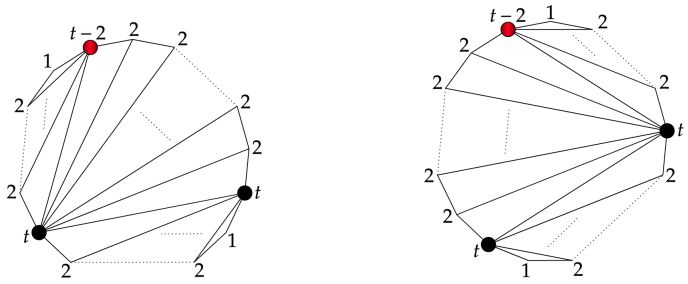


Figure: Basic wormhole triangulations with 3 weights from \mathcal{P}_0 and \mathcal{P}_2 .

Example of basic wormhole triangulations: changing the frame.

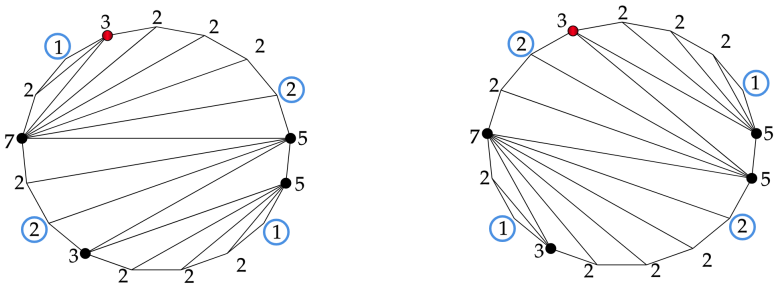


Figure: Basic wormhole triangulations with 5 weights frame from \mathcal{P}_0 and \mathcal{P}_2 .

They define the extended zero continued fractions

$$0 = [1, 2, 7, 2, 2, 3, 2, 2, 2, 2, 1, 5, 5, 2, 2, 2, 2 \mid 3],$$

$$0 = [2, 2, 7, 2, 1, 3, 2, 2, 2, 2, 2, 5, 5, 1, 2, 2, 2 \mid 3].$$

It is clear that they are WW-decompositions of the chain

$$\frac{m}{m - q} = [2, 2, 7, 2, 2, 3, 2, 2, 2, 2, 2, 5, 5, 2, 2, 2, 2]$$

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Once we have the triangulation from our main theorem, we can change coherently the hidden index in both triangulations. **The only restriction is that it cannot be one of the positions of the index 1 vertices in each triangulation.**

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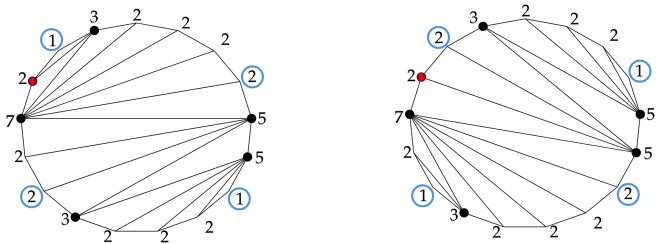


Figure: Basic wormhole triangulations with 5 weights frame from \mathcal{P}_0 and \mathcal{P}_2 .

They define the extended zero continued fractions

$$[7, 2, 2, 3, 2, 2, 2, 1, 5, 5, 2, 2, 2, 2, 3, 1 \mid 2],$$

$$[7, 2, 1, 3, 2, 2, 2, 2, 5, 5, 1, 2, 2, 2, 3, 2 \mid 2].$$

It is clear that they are WW-decompositions of the chain

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